Descriptive Set Theory HW 6

Thomas Dean

Problem 1. Show the following:

1. LO is a closed subset of $2^{\mathbb{N}^2}$ and WO is co-analytic.
2. Prove that WO is actually $\mathbf{\Pi}^1_1$-complete.

Solution.

1. Given $x \in 2^{\mathbb{N}^2}$, notice that $n < m$ is a clopen condition because it corresponds with checking $x(n, m) = 1$. When you write out the definition of a linear order, one notices that we're just universally quantifying a bunch of clopen expressions like $(n < m \wedge m < k) \to n < k$. Since universal quantification is the same as taking intersections, we get at the end of the day that LO is closed. For WO, notice that $x \in \text{WO} \Leftrightarrow x \in \text{LO}$ and $(\forall y \in \mathbb{N}^{\mathbb{N}})(\exists n \in \mathbb{N})\ x(y(n+1), y(n)) \neq 1$. In other words, x is a linear order that doesn't have an infinite descending sequence. This is co-analytic because both of the conjuncts are co-analytic (we're making one universal quantification over a real).

2. It's enough to define a continuous map $f\colon Tr \to LO$ such that $f^{-1}[WO] = WF$ because we know that WF is $\mathbf{\Pi}^1_1$-complete. We recall the Kleene-Brouwer ordering $<$ on trees, which turns out to be a linear order, and is well-founded on a tree T exactly when T doesn't have a branch. Identify ω with $\omega^{<\omega}$ by fixing an enumeration $b\colon \omega \to \omega^{<\omega}$. Define a function $f\colon Tr \to LO$ by letting

   $f(T)(s, t) = 1 \Leftrightarrow (s, t \in T$ and $s < t)$ or $(s \notin T, t \notin T, b^{-1}(s) < b^{-1}(t))$ or $(s \in T, t \notin T)$.

   In other words, we order the elements of T before the elements not in T, and we order the elements of T by the Kleene-Brouwer ordering. The elements not in T are ordered by where they come in our enumeration b. By definition it then follows that $f(T) \in LO$ for each $T \in Tr$. Further, it follows that T is well-founded exactly when $f(T) \in WO$ because we ordered T using the Kleene-Brouwer ordering and the enumeration has order type ω. It remains to check that f is continuous. But, this is true because f is computable using the elements $T \in 2^{\omega^{<\omega}}$ and b as oracles (by Lemma 3.11 in Marker's DST notes, this is equivalent to being continuous).

⋆

Problem 2. Let E be an equivalence relation on a Polish space X. Prove that $\mathrm{id}(2^{\mathbb{N}}) \leq_B E$ iff $\mathrm{id}(2^{\mathbb{N}}) \sqsubseteq_B E$ iff $\mathrm{id}(2^{\mathbb{N}}) \sqsubseteq_c E$.

Solution. The converse direction of $\mathrm{id}(2^{\mathbb{N}}) \sqsubseteq_B E$ iff $\mathrm{id}(2^{\mathbb{N}}) \sqsubseteq_c E$ is clear, and the forward direction holds because we can take a witness $f\colon 2^{\mathbb{N}} \to X$ to $\mathrm{id}(2^{\mathbb{N}}) \sqsubseteq_B E$ and refine the topology τ on $2^{\mathbb{N}}$ to another Polish topology τ^* with the same Borel sets that makes f continuous. Depending on what is meant by $\mathrm{id}(2^{\mathbb{N}})$, we might not be done, because we just changed topologies to τ^*. However, since $(2^{\mathbb{N}}, \tau^*)$ is an uncountable Polish space, it contains a homeomorphic copy of $(2^{\mathbb{N}}, \tau)$. If $g\colon (2^{\mathbb{N}}, \tau) \to (2^{\mathbb{N}}, \tau^*)$ is an embedding that witnesses this, then $f \circ g\colon (2^{\mathbb{N}}, \tau) \to X$ witnesses that $\mathrm{id}(2^{\mathbb{N}}) \sqsubseteq_c E$.

For the other iff, the one direction is clear. Assume that $\mathrm{id}(2^{\mathbb{N}}) \leq_B E$ and let $f\colon 2^{\mathbb{N}} \to X$ witness this. If $f(x) = f(y)$, then $f(x)Ef(y)$, and so $x = y$ by assumption on f. This witnesses that $\mathrm{id}(2^{\mathbb{N}}) \sqsubseteq_B E$. ⋆

Problem 3. Fill in the details in the proof of Mycielskis theorem.

Solution. Given a meager equivalence relation E on a Polish space X, write $E = \cup_n F_n$, where $n \geq 1$ and F_n are increasing and nowhere dense. Without loss of generality we can assume that $(x, y) \in F$ implies that $(y, x) \in F$ because the map $(x, y) \mapsto (y, x)$ is a homeomorphism. We construct a Cantor scheme $(U_s)_{s \in 2^{<\omega}}$ to satisfy the conditions mentioned in the statement of problem 81:

Set $U_\varnothing = X$. Assume that we've defined pairwise disjoint U_s for each $s \in 2^n$ such that $(U_s \times U_t) \cap F_n = \varnothing$ for distinct s and t with height n. We construct U_s inductively for $s \in 2^{n+1}$ as follows: first, for each $s \in 2^n$, let $V_{s^\frown i} \subseteq \overline{U_s}$ for $i = 0, 1$ be pairwise disjoint and small enough so that we'll have vanishing diameter at the end of the day. We can do this because X is perfect (otherwise $\{(x, x)\}$ is non-meager and $\{(x, x)\} \subseteq E$). Next, using the lexicographical ordering $<$ on 2^{n+1}, what we do is recursively choose U_s by forcing that $(U_s \times U_t) \cap F_{n+1} = \varnothing$ for any other t. We do this by iteratively applying the fact that, for any open U, V, there's $U' \subseteq U$ and $V' \subseteq V$ such that $(U' \times V') \cap F_{n+1} = \varnothing$. This is because F_{n+1} is nowhere dense. The construction gets a bit hairy because you have to keep track of what open sets you've been applying the above result to, so let me know if you want to talk about it. ⋆

Problem 4. Let X be a Polish space and E be an equivalence relation on X generated by a countable family $\{B_n\}_{n\in\mathbb{N}}$ of Borel sets. Prove that a Borel set $B \subseteq X$ is E-invariant iff it belongs to the σ-algebra generated by $\{B_n\}_{n\in\mathbb{N}}$.

Solution. For notation, let σ denote the σ-algebra generated by $\{B_n\}_{n\in\mathbb{N}}$.

For the forward direction, fix an invariant B and define $f\colon X \to 2^\omega$ by $f(x) = y \Leftrightarrow (\forall n)(y(n) = 1 \leftrightarrow x \in B_n)$. So, $f(x)$ codes exactly which B_n that x is in. This is Borel map because each B_n is Borel and we are taking countable intersections.

Next, observe that it's not too hard to show that $f^{-1}(C) \in \sigma$ for any Borel set $C \subseteq 2^\omega$. This follows because $\{C \subseteq 2^\omega \colon f^{-1}(C) \in \sigma\}$ is a σ-algebra containing the open sets. Now, let $A = f"B$. Because B is invariant, we get that $f^{-1}(A) = B$ by the definition of E. So, by problem 60, we get that there's a Borel $A' \subseteq 2^\omega$ such that $A = f"B = A' \cap f"X$. This implies that $f^{-1}(A') = B$, yielding $B \in \sigma$.

For the other direction, let $\mathcal{I}$ be the set of all $B \in \sigma$ such that B is E-invariant. For each B_n, observe that if there's a $b \in B_n$ such that xEb, then $x \in B_n$ by definition of E. So, $\mathcal{I}$ contains all elements of $\{B_n\}_{n\in\mathbb{N}}$. It's also not hard to check that $\mathcal{I}$ is a σ-algebra, which would imply that each element of σ is E-invariant, as desired.

For example, if B is invariant and $x \in [B^c] \cap B$, then there's a $b \in B^c$ such that xEb. This implies that $b \in [B] = B$. This contradicts that $b \notin B$. So $[B^c] \subseteq B^c \subseteq [B^c]$. ⋆

Problem 5. Prisoners and hats ($\mathbb{E}_0$ version)

Solution. This is the one where all the prisoners at once shout out what they think their hat color is. First, we identify the color blue with 0 and red with 1. The night before the execution, the prisoners use AC to choose an element f out of each equivalence class of $\mathbb{E}_0$, agreeing on which representatives they decided to choose. The day of the execution, once they're lined up, they observe that their hat colors induce a corresponding element x of 2^ω. Because they can all see each other, they know where they are in the line-up and therefore can determine which f they chose is $\mathbb{E}_0$-equivalent to x. Because the prisoner p_n knows all of the digits of x besides $x(n)$, prisoner p_n will guess $f(n)$ for the value of $x(n)$. Since two sequences are $\mathbb{E}_0$-equivalent when they agree on a tail end, we'll have that cofinitely many prisoners are saved. ⋆

Problem 6. Let $S \subseteq 2^{<\mathbb{N}}$.

1. If S contains at most one element of each length, then $\mathcal{G}_S$ is acyclic.

2. If S contains at least one element of each length, then $E_{\mathcal{G}_S} = \mathbb{E}_0$.

Solution.

1. Assume for contradiction that there is a cycle of length $n > 1$ (with no repeating vertex and $x_n = x_0$) and consider the longest $s \in S$ associated with its edges. Say that $x_i = s^\frown 1^\frown x$ and $x_{i+1} = s^\frown 0^\frown x$. where $0 \leq i < n$. Since $s \in S$ is the longest sequence associated with its edges, at no point will we flip the digits of x. If $x_0(|s|) = 1$, then at some point we'd have to flip the 0 in the $|s|$-th place back to a 1. Since s is the longest element of S associated with its edges and S contains at most one element of each length, there must then be a $k > i$ such that $x_k = s^\frown 1^\frown x$, contradicting that we don't have a repeating vertex. Similarly, if $x_0(|s|) = 0$, then there'd have to be a $k < i$ such that $x_k = s^\frown 0^\frown x$, also contracting that we don't have a repeating vertex.

2. We already know that $E_{\mathcal{G}_S} \subseteq \mathbb{E}_0$. To show the other direction, we show by induction that, for any $s, t \in 2^n$ and $x \in 2^\omega$, there's a $\mathcal{G}_0$ path connecting $s^\frown x$ and $t^\frown x$. This implies that any two sequences that agree on a tail end must be the same connected component of $\mathcal{G}_0$. Because S must contain an element of length 0, we must have that $\varnothing \in S$. This implies that the base case $n = 0$ holds. Now, assume for any $s, t \in 2^n$ and $x \in 2^\omega$, there's a $\mathcal{G}_0$ path connecting $s^\frown x$ and $t^\frown x$. Let $s, t \in 2^{n+1}$ and $x \in 2^\omega$. If $s(n) = t(n)$, then we can appeal to the induction hypothesis and we win. So assume without loss of generality that $s(n) = 1$ and $t(n) = 0$. Let s_n and t_n denote the restrictions of s and t to domain n. Fix a $k \in S$ that has length n. Then, the induction hypothesis implies that there's a $\mathcal{G}_0$ path connecting $s^\frown x = s_n^\frown 1^\frown x$ to $k^\frown 1^\frown x$. By definition of $\mathcal{G}_0$, since $k \in S$, we have that $k^\frown 1^\frown x \mathcal{G}_0 k^\frown 0^\frown x$. By the induction hypothesis again, we get that there's a path connecting $k^\frown 0^\frown x$ to $t_n^\frown 0^\frown x = t^\frown x$, completing the induction.

⋆

Problem 7. Prisoners and hats ($\mathcal{G}_0$ version)

Solution. Like before, the night before the execution, the prisoners use AC to choose an element f out of each equivalence class of $\mathbb{E}_0$, agreeing on which representatives they decided to choose. The day of the execution, with all the prisoners lined up, let $x \in 2^\omega$ denote the binary sequence induced from the hat colors (with blue being 0 and red being 1). Because they can all see each other, they know where they are in the line-up and therefore can determine which f the sequence x is $\mathbb{E}_0$-equivalent to, where f is the one they all previously agreed on. Prisoner p_0 counts the number of times that x and f disagree after the first digit (because p_0 doesn't know what $x(0)$ is) and guesses 0 if it's an even number of disagreements. Otherwise he guesses 1. Without loss of generality, let's say that p_n guesses 0. Because they eventually agree, there will necessarily be such a number. There's a 50-50 chance p_0 guesses the right answer. For any $n \geq 1$, p_n counts the number of times that x and f disagree after the first digit, not including the n-th digit of x and f (because by assumption p_n only doesn't know what $x(n)$ is). If p_n counts an even number of disagreements, then $x(n) = f(n)$, or else p_0 would have said they disagreed an odd number of times. In this case, p_n guesses whatever $f(n)$ and is set free. Otherwise, p_n guesses $1 - f(n)$. ⋆